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Quasinormal subgroups of finite p-groups

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Abstract. The distribution of quasinormal subgroups within a group is not particularly well understood. Maximal ones are clearly normal, but little is known about minimal ones or about maximal chains. The study of these subgroups in finite groups quickly reduces to p-groups. Also within an abelian quasinormal subgroup, others (quasinormal in the whole group) abound. But in non-abelian quasinormal subgroups, the existence of others can be dramatically rare.

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1 Introduction and development of the theory

A subgroup Q of a group G is said to be *quasinormal* in G if $QH = HQ$ (the subgroup generated by Q and H) for every subgroup H of G. In this situation we write Q *qn* G. (The term *permutable* has also been used on occasions; but then the natural implication, when referring to two permutable subgroups, is that they simply permute with each other under multiplication.) Clearly normal subgroups are quasinormal, but not conversely in general. However, Ore [11] proved that *quasinormal subgroups of finite groups are subnormal*; and in separate unpublished work, Napolitani and Stonehewer showed that *quasinormal subgroups of infinite groups are ascendant in at most* $\omega + 1$ *steps.* For most of what follows, however, we shall restrict ourselves to finite groups.

The simplest examples of non-normal quasinormal subgroups are to be found in the non-abelian groups of order p^3 and exponent p^2 (for an odd prime p); and there they are of course abelian of order p. Indeed Itô and Szép [7] proved that *a quasinormal subgroup of a finite group is always nilpotent modulo its normal core.* In 1967 and 1968, respectively, Thompson [16] and Nakamura [9] gave core-free examples of nilpotency class 2. Then in 1971 and 1973, Bradway, Gross and Scott [2] and Stonehewer [13] showed that *any nilpotency class is possible.* Following this (in [14]), examples of core-free quasinormal subgroups were constructed *with derived length d, for any positive integer d.* But by this time a significant improvement on the Itô - Szép result had been established by Maier and Schmid [8], viz.

if Q is a quasinormal subgroup of a finite group G, with normal core Q_G, then Q/Q_G lies in the hypercentre of G/Q_G.
It follows easily from this result that the problems associated with quasinormal subgroups of finite groups reduce quickly to p-groups. For, if Q is quasinormal in G and $Q_G = 1$, then it is easy to show that each Sylow p-subgroup P of Q is also quasinormal in G. (See [12], Lemma 6.2.16.) Also, by [8], each p'-element of G commutes with P elementwise. Therefore if S is a Sylow p-subgroup of G, then

P is quasinormal in S

and the complexities of Q's embedding in G reduce to those of P's embedding in S.

It is worth pointing out that there is a very good reason for studying quasinormal subgroups in finite p-groups, apart from their curiosity value. This relates to *modular* subgroups. Recall that a subgroup M of a group G is *modular* if, for each subgroup U of G, the map $H \mapsto U \cap H$ is a lattice isomorphism from $[\langle U, M\rangle/M]$ to $[U/(U \cap M)]$. Since modular subgroups are invariant under lattice isomorphisms, and since the quasinormal and modular subgroups of finite p-groups coincide ([12], Theorem 5.1.1), it follows that in finite p-groups

quasinormal subgroups are invariant under lattice isomorphisms.

Of course normal subgroups do *not* satisfy this property.

2 Abelian quasinormal subgroups

In trying to understand more about quasinormal subgroups, it is surely natural to begin with the abelian ones, even the cyclic ones. Indeed if Q is a cyclic quasinormal subgroup of a group G, then

every subgroup of Q is also quasinormal in G.

(See [12], Lemma 5.2.11.) This result is true for all groups G, not just the finite ones. Following a conjecture by Busetto and Napolitani, much more was discovered about the cyclic case by Cossey and Stonehewer in [3]:-

If Q is a cyclic quasinormal subgroup of odd order in a finite group G, then $[Q, G]$ is abelian and Q acts by conjugation on $[Q, G]$ as power automorphisms. Thus the normal closure Q^G is abelian-by-cyclic.
A key situation in establishing this result showed that each element of $[Q, G]$ has the form $[q, g]$, with $q \in Q$ and $g \in G$. Note that Q does not have to be core-free here. The case when Q has even order is considerably more complicated and is

dealt with in [4] and [5]. There are examples where $[Q, G]$ is *not* abelian, but it is always nilpotent of class at most 2.

In moving from cyclic to abelian quasinormal subgroups Q, it is clear that not all subgroups of Q will be quasinormal. But there are surprisingly many of them:-

Let Q be an abelian quasinormal subgroup of G (finite or infinite). Then Q^n is quasinormal in G, provided n is odd or divisible by 4.

(See [15].) Again it is not necessary for Q to be core-free here. But there are examples with G finite where Q^2 is not quasinormal. However, now that chains of quasinormal subgroups are beginning to appear, it is natural to ask if, given Q *qn* G, a finite p-group, there are maximal chains of quasinormal subgroups of G, passing through Q, that are composition series of G. We may even ask if *all* maximal chains of quasinormal subgroups in a finite p-group have to be composition series. We shall see below that the answers here are "no". But for abelian Q, a lot can be said in a positive direction.

Recall that Q *qn* G if and only if $QX = XQ$ for all cyclic subgroups X of G. Thus a significant partial stage, on the way to understanding more about quasinormal subgroups, is to be able to make statements about quasinormal subgroups Q of groups G of the form

$$G = QX, \tag{1}$$

where X is cyclic. Nakamura showed in [10] that in this situation when G is a finite p-group, Q always contains a quasinormal subgroup of G of order p. For the moment, we shall assume that (1) holds. Moreover we shall assume that

$$G \textit{ is a finite } p\textit{-group and } Q_G = 1.$$

Then clearly X contains a non-trivial normal subgroup of G and so

$$\Omega_1(X) \leqslant Z(G).$$

Here $Z(G)$ is the centre of G and $\Omega_1(G)$ denotes the subgroup generated by the elements of order p. More generally $\Omega_i(G)$ will denote the subgroup generated by the elements of order at most p^i. Then the following result, due to Cossey and Stonehewer, will appear in the Journal of Algebra in the volume dedicated to the memory of Karl Gruenberg:-

Theorem 1. *Let Q qn $G = QX$, with G a finite p-group, $Q_G = 1$, Q abelian and X cyclic. Then*

(a) *$W_i = \Omega_i(Q)$ qn G, for all $i \geqslant 1$;*

(b) *there exists a composition series of G, passing through the W_i's, in which every subgroup is quasinormal in G; and*

(c) *if p is odd, then there is a composition series of G, passing through the p^i-th powers of Q, in which every subgroup is quasinormal in G.*

Removing the hypothesis (1) is not easy. All that we can say to date is the following (see [6]):-

Theorem 2. *If Q is a quasinormal subgroup of order p^2 in a finite p-group G (with p an odd prime), then there is a quasinormal subgroup of G of order p lying in Q.*

Unfortunately there is nothing canonical about this subgroup of order p, and its existence was established only by an exhaustive survey of all possibilities. Thus for our final section we shall revert to the hypothesis (1).

3 Non-abelian quasinormal subgroups

There is a universal embedding theorem for the situation (1), due to Berger and Gross [1]:-

Given a prime p and an integer $n \geqslant 1$, there exists a finite p-group

$$G = QX$$

such that

(i) *Q qn G, $Q_G = 1$ (so $Q \cap X = 1$) and $X = \langle x \rangle$ is cyclic of order p^n;*

(ii) *if Q^* qn $G^* = Q^*X^*$, a finite p-group, with $Q^*_{G^*} = 1$ and $X^* = \langle x^* \rangle$ is cyclic of order p^n, then G^* embeds in G uniquely such that Q^* embeds in Q and x^* maps to x.*

The group G has exponent p^n and Q has exponent p^{n-1}. Moreover

$$\Omega_1(G) = \Omega_1(Q) \times \Omega_1(X)$$

is elementary abelian and an indecomposable X-module. Let $G_n = G$. Then $G_n/\Omega_1(G_n) \cong G_{n-1}$. Also $\Omega_i(G_n)$ has exponent p^i. Berger and Gross define G_n as a permutation group on the integers modulo p^n and Q is the stabiliser of $\{0\}$.

Assume (for simplicity) that p is odd. In Canberra in 2007, Cossey, Stonehewer and Zacher began to study these groups G_n for small values of n. In Warwick in 2009, Cossey and Stonehewer have continued this work for arbitrary n and a succession of results has been obtained, giving a fairly complete picture, with a somewhat surprising conclusion.

The first non-trivial case is when $n = 2$. Here Q is elementary abelian of rank $p - 2$. Also $\Omega_1(G) = Q \times \Omega_1(X)$ is a uniserial X-module. Thus there is a unique chief series of G between $\Omega_1(X)$ and $\Omega_1(G)$ and *the intersections of its terms with Q are precisely the quasinormal subgroups of G lying in Q.* So they form part of a composition series of G passing through Q. This is a special case of Theorem 1 above.

Now suppose that $n = 3$. Here $\Omega_1(G) = \Omega_1(Q) \times \Omega_1(X)$ is an indecomposable X-module of rank $p(p-1) = r+1$, say. Again *the quasinormal subgroups of* G, *of exponent* p *and lying in* Q, *are precisely the non-trivial intersections with* Q *of the terms of the unique chief series of* G *between* $\Omega_1(G)$ *and* $\Omega_1(X)$. Denote these intersections by

$$\Omega_1(Q) = W_r > W_{r-1} > \cdot \cdot \cdot > W_1 \ (> W_0 = 1).$$

Modulo $\Omega_1(G)$, $\Omega_2(G)$ is an indecomposable X-module of rank $p-1$. Let $X_2 = \Omega_2(X)$ and let Q_1 be a quasinormal subgroup of G, of exponent p^2, lying in Q. Then

(i) $\Omega_1(Q_1) = W_i$, *some* $i \geqslant r - p$; *and*

(ii) Q_1X_2 *modulo* $\Omega_1(G)$ *is an* X*-submodule of* $\Omega_2(G)/\Omega_1(G)$.

Conversely, for any i, j *with* $r \geqslant i \geqslant r-p$ *and* $p-1 \geqslant j \geqslant 2$, *there is a quasinormal subgroup* Q_1 *of* G, *of exponent* p^2 *lying in* Q, *with* $\Omega_1(Q_1) = W_i$ *and* Q_1X_2 *modulo* $\Omega_1(G)$ *the* X*-submodule of* $\Omega_2(G)/\Omega_1(G)$ *of rank* j.

It follows that there are maximal chains of quasinormal subgroups of G, passing through Q, which are composition series of G.

Next we consider $G = G_n$ *for* $n = 4$. Here $\Omega_1(G)$ has rank $p^2(p-1) = s+1$, say. Again *the quasinormal subgroups of* G, *of exponent* p *lying in* Q, *are the non-trivial intersections with* Q *of the terms of the unique chief series of* G *between* $\Omega_1(G)$ *and* $\Omega_1(X)$, *but only those of rank* $\leqslant p^2 - 1$. If Q_1 is a quasinormal subgroup of G, of exponent p^2 lying in Q, then $\Omega_1(Q_1)X_1$ *is an* X*-submodule of* $\Omega_1(G)$ *of rank* $i+1$, *with* $i \geqslant s - p(p-1)$; *and* Q_1X_2 *modulo* $\Omega_1(G)$ *is an* X*-submodule of* $\Omega_2(G)/\Omega_1(G)$ *of rank* $j+1$, *with* $j \geqslant p-2$. *Moreover there are quasinormal subgroups* Q_1 *of this form for each of the above values of* i *and* j. But now we see that we have a 'gap' in a maximal chain of quasinormal subgroups passing through Q. Indeed the largest quasinormal subgroup of exponent p has rank p^2-1, and the smallest of exponent p^2 is elementary abelian of rank $p(p-1)^2 - 1$ extended by an elementary abelian group of rank $p-2$. The former is contained in the latter and has index

$$p^{p^2(p-3)+2(p-1)}.$$

So *there is no composition series of* G ($=G_4$) *passing through* Q *and consisting of quasinormal subgroups of* G.

How big can this gap be in general? If H is a quasinormal subgroup of G (a finite p-group) and K is a quasinormal subgroup of G maximal subject to being properly contained in H, then clearly $K \lhd H$. But is H/K restricted in some way? The answer is 'no'! Indeed G_5 has no quasinormal subgroups of exponent p^2 lying in Q. In fact G_n, for $n \geqslant 5$, has no quasinormal subgroups of exponent p^2 lying in Q. Since $G_5 \cong G_6/\Omega_1(G_6)$, G_6 has no quasinormal subgroups of exponent p^3 lying in Q; and so on. The situation is as follows:-

Theorem 3. *The only non-trivial quasinormal subgroups of G_n ($n \geqslant 2$), lying in Q, have exponent*

$$p,\ p^{n-2} \text{ and } p^{n-1}.$$

Thus there is a 'black hole' between exponent p and exponent p^{n-2}. To sum up, let Q_1 be a quasinormal subgroup of G_n lying in Q and let $X_i = \Omega_i(X)$, all i. Then for each i with $p^i \leqslant$ exponent of Q_1,

$\Omega_i(Q_1)X_i$ modulo $\Omega_{i-1}(G)$ is a submodule of the indecomposable X-module $\Omega_i(G)/\Omega_{i-1}(G)$.

If the rank of this submodule is r_i, then r_i is restricted to a known range of values. Moreover, there is a quasinormal subgroup Q_1 for each choice of values of r_i within each range. The somewhat lengthy proofs will appear elsewhere.

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